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Orbital integrals

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This is my attempt to understand the theory of orbital integrals on reductive groups over local fields. I should confess right off that I follow here a variant of the motto 'learn by teaching'. In truth, I was tempted to make my title 'Orbital integrals for Dummies', but decided that this title was one for me as reader rather than for me as author. My initial motivation in beginning this project was that the extant expository literature seems to be quite disorganized, and certainly not suitable for Dummies. The need for a coherent account is great. On the one hand, now that the Fundamental Lemma has been proved, one might expect a better understanding of what is difficult and what is easy. On the other, the problems involved in proving Langlands' functoriality conjectures seem a bit more exposed to attack.

There are roughly speaking three threads to the topic: (1) *Orbital integrals for real groups.* Here there are two subtopics of major interest: (a) the relationship with characters of irreducible representations, most clearly stated in [Barbasch-Vogan:1980]; (b) the 'jump conditions' describing the nature of the singularities of orbital integrals near points where different classes of tori meet. (2) *Orbital integrals for p-adic groups.* Here the connection with fixed points on the Brutah-Tits building as well as on certain 'Springer fibres' is one main subtopic, and another is the definition of 'transfer factors' by Galois cohomology. These are the p-adic analogue of the jump conditions on real groups, but are much more subtle. (3) *The theory of Igusa*, which investigates very generally the behaviour of integrals parametrized by a base variety. Interesting and serious questions involving algebraic geometry of varieties over non-algebraically closed fields arise.

The theory for real groups is primarily the work of Harish-Chandra, Robert Langlands, and Diana Shelstad, but with later additions by David Vogan, Dan Barbasch, Abderrazak Bouaziz, Jim Arthur and a few others. This is largely concerned with the behaviour of solutions to certain differential equations. This subject is reasonably complete.

That for p-adic fields is due to a much larger number of people, including Harish-Chandra, Langlands, and Shelstad, and culminating recently with the proof by Ngô Bau Châu of the Fundamental Lemma. Others whose contributions so far have been extremely important are Labesse, Kottwitz, Waldspurger, Rogawski, Hales, and Laumon. This list should make it clear how difficult the task of summarizing all accomplishments would be. In any case, although much has been done on this topic there are still, as far as I can see, many mysteries.

The p-adic case is particularly interesting because it makes clear that in time the theory should have astonishing arithmetic consequences.

The connection between Igusa's theory and orbital integrals has had so far a somewhat unexploited career due mostly to Langlands and Shelstad. However, in my opinion it provides the best introduction to the subject, and that is why I shall start out with it. Orbital integrals are examples of something very general, in which a group acts on a manifold and one wants to understand the representation of the group on various spaces of functions on the manifold, in particular what is in some sense the quotient of the manifold by the group. If the group is compact, this quotient is analytically rather simple, but interesting phenomena arise when G is not compact. I shall begin with orthogonal groups $G = \mathrm{SO}(Q)$ acting on quadratic spaces. Those for which G is compact are easy analytically, but nonetheless suggestive. Those in which G is non-compact are more interesting. Although these examples, with one notable exception, have little to do directly with orbital integrals, they should give some idea of the problems one faces, and in a situation where one's intuition is valuable.

Contents

Part I. Orbital integrals

1. Compact groups

For the moment, let G be be a compact semi-simple group, T a maximal torus. Under the adjoint action of T, the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ decomposes into the direct sum of T and one-dimensional eigenspaces $\mathfrak{g}_\chi$, where χ is a character $T \to \mathbb{S}$, the group of complex numbers z with $z\overline{z} = 1$. The characters appearing are the roots of $\mathfrak{g}$ with respect to T. If $\chi(t) \neq 1$ for all roots χ then t is called regular, and in this case t is its own centralizer. The corresponding orbital integral is the distribution

$$\langle O_t, f\rangle = \int_{T\backslash G} f(g^{-1}tg)\, dg\,.$$

2. The group $\mathrm{SL}_2(\mathbb{R})$

3. References

All papers by Arthur, Langlands, and Shelstad are avialable for download from the Internet, without access restrictions.

1. James G. Arthur, 'Germ expansions for real groups', preprint, 2006.

This has not been formally published, but is available at the Clay Institute Archive of Arthur's works. It is basically a continuation of Bouaziz' articles. Although suggestive, it is incomplete. A mote satisfactory account would presumably take into account the natural filtration of the infinitesinal neighbourhood of the unipotent variety.

2. Dan Barbasch and David Vogan, 'The local structure of characters', *Journal of Functional Analysis* **37** (1980), 27–55.

One of the principal facts about orbital integrals is that their Fourier transforms on the group G are closely related to characters of irreducible representations. This manifests itself on the Lie algebra in the fact that the asymptotic behaviour of characters near the identity on G is described by the Fourier transforms of the Lie algebra analogues of orbital integrals.

3. Abderrazak Bouaziz, 'Intégrales orbitales sur les groupes de Lie réductifs', *Inventiones Mathematicae* **115** (1994), 163–207.

4. ——, 'Intégrales orbitales sur les algèbres de Lie réductives', *Annales scientifiques de l'École Normale Supérieure* **27** (1994), 573–609.

5. I. M. Gelfand and G. E. Shilov, **Properties and operations**, volume I of **Generalized functions**, Academic Press, 1964.

§4.5 is *Integrals of an infinitely differentiable function over a surface given by* $G = c$. One example it considers is that of the hyperbolic plane.

6. I. M. Gelfand, M. I. Graev, and I. I. Piatetski-Shapiro, **Representation theory and automorphic forms**, W. B. Saunders Company, 1969.

Section 5 of Chapter 1 is about the trace formula for quotients of $\mathrm{SL}_2(\mathbb{R})$ and applications.

7. M. Goresky, R. E. Kottwitz, and R. MacPherson, a paper about affine Springer fibres, *Duke Mathematical Journal*.

This transfers the question of fixed points on the building to one about fixed points on certain algebraic varieties defined over finite fields.

8. Tom Hales, 'Orbital integrals on U(3)', in **The zeta functions of Picard modular surfaces**, published by the CRM in Montréal, 1992.

There are several papers in this volume, by Hales, Langlands, Kottwitz, Blasius, and Rogawski, doing for SU(2,1) what Langlands and Kottwitz had done for SL(3) and GL(3). The one by Hales is particularly good at giving some idea of what endoscopy is all about.

9. Harish-Chandra, 'A formula for semi-simple Lie groups', *American Journal of Mathematics* **79** (1957), 733–760.

Sooner or later you will have to look at Harish-Chandra's papers. For a very rough guide to what you'll find, skim Varadarajan's introduction to the collected works, which starts on p. xxxvi of Volume I.

10. ——, 'Some results on an invariant integral on a semi-simple Lie algebra', *Annals of Mathematics* **80** (1964), 551–593.

11. ——, 'Discrete series for semi-simple groups II', *Acta Mathematica* **116** (1966), 1–111.

This paper, despite its title, contains an introduction to Harish-Chandra's 'Schwartz' space $\mathcal{G}(G)$.

12. Jun-Ichi Igusa, **Lectures on forms of higher degree**, Tata Institute for Fundamental Research, 1972.

An early version posted at TIFR was full of typographical errors, but a scan of the original publication has now been made available:

`www.math.tifr.res.in/ publ/ln/igusa.pdf`

13. Pierre Jeanquartier, 'Transformation de Mellin et développements asymptotiques', *Enseign. Mathématique* **25** (1979), numbers 3–4, 285–308.

14. ——, 'Développement asymptotique de la distribution de Dirac attachée à une fonction analytique', *Comptes Rendus de l'Academie Scientifique de Paris* **271** (1970), 1159–1161.

15. Masaki Kashiwara and Takahiro Kawai, 'Second-microlocalization and asymptotic expansions', 21–76 in **Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory**, edited by D. Iagolnitzer, of the series *Lecture Notes in Physics* **126**, Springer-Verlag, 1980.

16. Kottwitz, Robert Edward, ' Orbital integrals on GL_3', *American Journal of Mathematics* **102** (1980), 327–384.

This and the related paper on $\mathrm{SL}(3)$ are the beginning of the analysis of orbital integrals in terms of fixed points on the Bruhat-Tits building. Although this also proved to be a dead end, it eventually led to the papers by Goresky, Kottwitz, and MacPherson about fixed points on affine Springer fibres, which led in turn to the final proof of the Fundamental Lemma.

17. ——, 'Unstable orbital integrals on SL_3', *Duke Mathematical Journal* **48** (1981), 649–664.

18. ——, 'Harmonic analysis on reductive p-adic groups and Lie algebra', 393–522 in **Harmonic analysis, the trace formula, and Shimura varieties**, *Clay Mathematics Proceedings* **4**, AMS, 2005.

19. Jean-Pierre Labesse and Robert P. Langlands, 'L-indistinguishability for $\mathrm{SL}(2)$', *Canadian Journal of Mathematics* **XXXI** (1979), xxx–yyy.

This contains a succinct verification of transfer from SL_2 to its tori. The p-adic case is done in a thorough if condensed argument, but for $\mathrm{SL}_2(\mathbb{R})$ they just refer to Harish-Chandra.

20. R. P. Langlands, 'Dimension of spaces of automorphic forms', in the account of the proceedings of the 1965 Boulder conference on automorphic forms. The entire publication can be downloaded from the AMS.

This is a short account of the longer paper with the same name. These generalize to other semi-simple groups the results found in Gelfand et al. about the discrete series of $\mathrm{SL}_2(\mathbb{R})$.

21. ——, **Base change for GL(2)**, *Annals of Mathematics Studies* **96**, Princeton University Press, 1980.

This contains the original characterization of orbital integrals on which Shelstad's Corvallis talk as well as Bouaziz' later descriptions are based. Unfortunately this discussion is hard to follow, partly because the notation is not quite conventional. This book also contains the first account of p-adic orbital integrals for the p-adic group GL_2 in terms of fixed points on the Bruhat-Tits building.

22. ——, 'Orbital integrals on forms of SL(3) I', *American Journal of Mathematics* **105** (1983), 465â^506.

23. ——, 'Remarks on Igusa theory and real orbital integrals', pp. 335–347 in **The zeta functions of Picard modular surfaces**, Centre de Recherches Mathématiques, Montréal, 1992.

This is one of a series comprising an aborted attempt to apply Igusa's theory to the computation of both real and p-adic orbital integrals. This work came to a halt, but my own opinion is that there is in this stuff much interesting material to explore. Unfortunately, these papers are are not easy to digest. This one, the only one dealing with real groups, is especially difficult.

24. —— and Diana Shelstad, 'On principal values on p-adic manifolds', pp. 250â^279 in **Lie group representations II**, *Lecture Notes in Mathematics* **1041**, 1984.

25. ——, 'Orbital integrals on forms of SL(3) II', *Canadian Journal of Mathematics* **41** (1989), 480–507.

26. Dragan Miličić, 'Notes on the Plancherel formula', preprint, 2012.

These notes are one of the best treatments of the Harish-Chandra space $\mathcal{C}(G)$ of $\mathrm{SL}_2(\mathbb{R})$.

27. Frank Olver et al., **NIST handbook of mathematical functions**, Cambridge University Press, 2010.

28. Ranga Rao, 'Orbital integrals in reductive groups', *Annals of Mathematics* **96** (1972), 505–510.

29. Diana Shelstad, 'Characters and inner forms of a quasi-split group over $\mathbb{R}$', *Compositio Mathemeticae* **39** (1979), 11–45.

This contains the characterization of orbital integrals of functions in the Harish-Chandra space $\mathcal{C}(G)$ that is used for $\mathrm{GL}_2(\mathbb{R})$ in Langlands' book **Base change**, and also in her Corvallis account of that. But it is hard to extract.

30. ——, 'A formula for regular unipotent germs', in **Orbites unipotentes et représentations II**, *Astérisque* **171–172** (1989), 275–277.

This illustrates that even apparently simple and natural questions can involve you in non-trivial matters.

31. ——, 'Orbital integrals for GL(2,R)', pp. 107–110 in **Automorphic forms, representations, and L-functions**, *Proceedings of Symposia in Pure Mathematics* **33**, AMS, 1979.

3 Lecture III: Orbital integrals 3.1 Integration on a p-adic group 3.2 Orbital integrals 3.3 Relation with afne Springer bers. 2 2 3 3 3 4 5 8 9 13 14 16 16 20 22 24 27 32 34 35 36 36 37 40. Research supported by the NSF grant DMS-1302071, the Packard Foundation and the PCMI.. Â©0000 (copyright holder). 1. 2 Lectures on Springer theories and orbital integrals. Orbital integrals are an important technical tool in the theory of automorphic forms, where they enter into the formulation of various trace formulas. References. Helgason, Sigurdur (1984), Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions, Academic Press, ISBN 0-12-338301-3. Basis of this page is in Wikipedia. Text is available under the CC BY-SA 3.0 Unported License. My questions concerns Definition 1.2 of an orbital integral in the paper Orbital integrals on General Linear Groups by Cluckers and Denef. I will recall the definition below, but my question is: how does Definition 1.2 relate to the usual definition of an orbital integral? By the latter I mean for given a reductive group G over K and a regular element γ of $G(K)$ with centralizer G_γ, the orbital integral of a These orbital integrals allow us to compute the Shalika germs at some â€œvery ellipticâ€ elements in terms of number of rational points on some quasi-finite covers of the Hessenberg varieties in [8] which are subvarieties of (partial) flag varieties.Â One may consider the orbital integral µg (f) = Ad(G)(g) f (x)dx = R f (Ad(g)x). Orbital integrals on the Lie algebra are defined in the same manG/CG (g) ner.