Linear Algebraic Groups: a Crash Course

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This is a collection of notes for three lectures, designed to introduce linear algebraic groups quickly in a course on Geometric Invariant Theory. There are several good introductory textbooks; in particular, the books by Humphreys [H], Springer [S], and Borel [B]. Here I merely distill some of the material from Humphreys and Springer.

1 Definitions

We'll work over a fixed algebraically closed base field k.

Definition 1.1 An **algebraic group** G is a group object in the category of varieties over k. That is, G is a group and a variety, and the maps

$$G \times G \to G \quad (g,h) \mapsto gh \qquad \text{and} \qquad G \to G \quad g \mapsto g^{-1}$$

are morphisms of varieties. (And there is a distinguished k-point $e \in G$, the identity.)

A **homomorphism** of algebraic groups is a group homomorphism that is also a map of varieties.

In schemey language, another way to say this is that the functor h_G : **Schemes** $\to$ **Sets** factors through **Groups**.

Definition 1.2 A **linear algebraic group** is an affine variety that is an algebraic group.

Example 1.3 The multiplicative group $\mathbb{G}_m = k^* = \operatorname{Spec} k[x, x^{-1}]$ is an algebraic group.

The coordinate ring $k[G]$ of a linear algebraic group is a (commutative) Hopf algebra: it comes with maps

$$\delta : k[G] \to k[G] \otimes k[G] = k[G \times G] \qquad \text{(comultiplication)},$$
$$c : k[G] \to k[G] \text{(antipode)},$$
$$\epsilon : k[G] \to k \text{(counit)},$$

corresponding to the multiplication, inverse, and unit maps, respectively.

Example 1.4 For $\mathbb{G}_m$, we have $\delta(x) = x \otimes x$, $c(x) = x^{-1}$, and $\epsilon(x) = 1$.

Exercise 1.5 Work out the maps for the additive group $\mathbb{G}_a = \operatorname{Spec} k[x]$.

A pleasant feature of the theory is that the most important examples (for now) are also the most familiar ones.

Example 1.6 The general linear group is a LAG, with $GL_n = \operatorname{Spec} k[x_{ij}]_{\det}$, as is any Zariski-closed subgroup of GL_n.

Example 1.7 Particular examples of closed subgroups that come up:

$\mathbb{B} \subset GL_n$, upper-triangular matrices ("Borel").

$\mathbb{U} \subset \mathbb{B} \subset GL_n$, strictly upper-triangular matrices, with 1's on diagonal (unipotent).

$\mathbb{D} \subset \mathbb{B} \subset GL_n$, diagonal matrices (maximal torus).

In fact, although we've defined "linear" to mean "affine", it turns out that *all* such groups are closed subgroups of GL_n. (This justifies the terminology.)

Proposition 1.8 *Every linear algebraic group can be embedded as a closed subgroup in some GL_n.*

To prove this, we'll need a couple more basic notions.

Definition 1.9 A **(rational) representation** of G on a k-vector space V is a homomorphism $G \to GL(V)$.

A representation is **irreducible** if there is no nontrivial proper G-stable subspace; that is, no W such that $0 \neq W \subsetneq V$ with $G \cdot W \subseteq W$.

One can talk about representations for infinite-dimensional V, but we'll always assume they're **locally finite**: for all $v \in W$, there is a G-stable, finite-dimensional subspace W with $v \in W \subseteq V$.

Example 1.10 The main (and for us, essentially only) example of this is the action of G on $k[G]$. (Or $k[X]$, when G acts on a variety X.) Here, for $f \in k[G]$, we have $g \cdot f$ defined by $(g \cdot f)(x) = f(g^{-1}x)$ for all $x \in G$. This is sometimes called the action by **left translation** on functions, and it works whenever G acts (on the left) on a variety.

There is also an action by **right translation** which is sometimes useful. (But note that it is still a left action!) Denoting this by r_g, we have $r_g \cdot f$ given by $(r_g \cdot f)(x) = f(xg)$.

The key thing here is that $k[G]$ is locally finite:

Lemma 1.11 *If $V \subset k[G]$ is a finite-dimensional subspace, then there is a finite-dimensional G-stable subspace W with $V \subseteq W \subseteq k[G]$. (In particular, $k[G]$ is locally finite.)*

Proof. It clearly suffices to treat the case where V is one-dimensional, say spanned by f. Write

$$\widetilde{\delta}(f) = \sum_i m_i \otimes f_i$$

in $k[G \times G]$, corresponding to the map $(g, h) \mapsto g^{-1}h$. Only finitely many terms appear, say $i = 1, \dots, n$. Then

$$(g \cdot f)(x) = f(g^{-1}x) = \sum_i m_i(g) f_i(x),$$

so

$$g \cdot f = \sum_i m_i(g) f_i$$

lies in the span of $f_1, \dots, f_n$. Therefore the space W spanned by

$$\{g \cdot f \mid g \in G\},$$

which is manifestly G-stable, is also finite-dimensional. ∎

We now prove the Proposition.

Proof. Take generators $f_1, \dots, f_n$ for $k[G]$. By the Lemma, we may assume they're a basis for a G-stable subspace. We'll produce an embedding $G \hookrightarrow GL_n$.

In fact, we have a map $k[GL_n] = k[x_{ij}]_{\det} \to k[G]$, as follows. Consider the right translation action. As in the lemma, there are elements $m_{ij} \in k[G]$ with

$$r_g \cdot f_i = \sum_j f_j \, m_{ij}(g).$$

Define the map by $x_{ij} \mapsto m_{ij}$.

Since

$$f_i(g) = f_i(eg) = \sum_j f_j(e) m_{ij}(g),$$

we see that $f_i = \sum_j f_j(e) m_{ij}$, and therefore the m_{ij} also generate $k[G]$. It follows that the map we defined is surjective, so it corresponds to a closed embedding of varieties. ■

This is all good culture, but many of the groups you encounter come automatically linearized. A major example is that of *diagonalizable groups*.

2 Diagonalizable groups and characters

The group of diagonal matrices $\mathbb{D}_n \subset GL_n$ is special in several ways. First, observe that

$$k[\mathbb{D}_n] \cong k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \cong k[\mathbb{Z}^n].$$

Definition 2.1 A **character** of an algebraic group G is a homomorphism $\chi : G \to \mathbb{G}_m = k^*$. The set of all characters forms an abelian group under pointwise multiplication, the **character group** of G, denoted $\mathrm{X}(G) = \mathrm{Hom}_{\text{alg. gp.}}(G, \mathbb{G}_m)$. (Warning: the group operation in $\mathrm{X}(G)$ is often written additively, so you may see the character $g \mapsto \chi_1(g)\chi_2(g)$ written as either $\chi_1 \cdot \chi_2$ or $\chi_1 + \chi_2$.)

Example 2.2 For $\mathbb{G}_m$, we have $\mathrm{X}(\mathbb{G}_m) = \mathbb{Z}$ canonically (up to choice of generator $1 \in \mathbb{Z}$), by sending the identity in $\mathrm{X}(G) = \mathrm{Hom}_{\text{alg. gp.}}(\mathbb{G}_m, \mathbb{G}_m)$ to $1 \in \mathbb{Z}$.

The character $\chi : z \mapsto z^n$ then corresponds to the integer n.

From the example, we see $\mathrm{X}(\mathbb{D}_n) \cong \mathbb{Z}^n$, and $k[\mathbb{D}_n] \cong k[\mathrm{X}(\mathbb{D}_n)]$. In other words, the group of diagonal matrices has lots of characters, enough to form a linear basis for all functions. Contrast with this with the case of a simple group like PGL_n, which has no nontrivial characters (since $\ker(\chi)$ would be a nontrivial normal subgroup).

Definition 2.3 A linear algebraic group is **diagonalizable** if it is isomorphic to a closed subgroup of some $\mathbb{D}_n$. A connected diagonalizable group is called a **torus**.

The key fact about diagonalizable groups is the following structure theorem:

Proposition 2.4 *For a linear algebraic group D, the following are equivalent:*

(1). D is diagonalizable

(2). $\mathrm{X}(D)$ is finitely generated, and $k[D] \cong k[\mathrm{X}(D)] := \bigoplus_{\chi \in \mathrm{X}(D)} k \cdot \chi$.

(3). Every rational representation of D is isomorphic to a direct sum of one-dimensional representations.

(4). D is isomorphic to $(k^*)^r \times A$, for some finite abelian group A.

Remark 2.5 In (3), the claim is that a representation V breaks up as $V = \bigoplus_\chi V_\chi$, where $V_\chi = \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in D\}$. Characters with $V_\chi \neq 0$ are called **weights**, and V_χ are called **weight spaces**.

Example 2.6 Take $T = (k^*)^2 = \mathbb{D}_2 \subset GL_2$, acting on 2×2 matrices $M_{2,2}$ by conjugation:

$$g \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1^{-1} & 0 \\ 0 & z_2 \end{pmatrix} = \begin{pmatrix} a & z_1 z_2^{-1} b \\ z_1^{-1} z_2 c & d \end{pmatrix}.$$

The group $\mathbb{D}_2$ has a basis of characters χ_1, χ_2, with $\chi_i(g) = z_i$, and the four-dimensional vector space $M_{2,2}$ breaks up as

$$M_{2,2} = \underbrace{k \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{weight } \chi_1 \chi_2^{-1}} \oplus \underbrace{k \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{weight } \chi_1^{-1} \chi_2} \oplus \underbrace{\left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}}_{\text{weight } 0}$$

Remark 2.7 Actually, there's some delicacy about which finite abelian groups A can occur in (4). The condition is that A should have no p-torsion if $\mathrm{char}(k) = p$.

Example 2.8 The diagonalizable group D with character group $\mathrm{X}(D) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has $k[D] \cong k[x, x^{-1}, y]/(y^2 - 1)$. So $D \cong \mathbb{G}_m \times \mu_2$. Note that if $\mathrm{char}(k) = 2$, this is a non-reduced group scheme (so not an algebraic group).

We now prove the proposition (see [S, §3]).

Proof. The implications (2)⇒(4)⇒(1)⇒(2) are easily verified. We'll show (2)⇒(3).

Let V be a finite-dimensional (rational) representation of D, corresponding to a homomorphism $\varphi : D \to GL(V)$. Choosing a basis for V, the map φ is given by

$$\varphi(g) = (a_{ij}(g)),$$

for some functions $a_{ij} \in k[D]$. By (2), we can write $a_{ij} = \sum_\chi c_{ij}^\chi \chi$ (with finitely many nonzero terms). Grouping these by characters, we can define matrices $A_\chi = (c_{ij}^\chi)$, and then we have

$$\varphi(g) = \sum_\chi \chi(g) A_\chi.$$

It's easy to see that the endomorphisms A_χ do not depend on the choice of basis.

We claim that A_χ is actually the projection on the weight space V_χ. To see this, we first show that $A_\chi \cdot A_\psi = \delta_{\chi,\psi} A_\chi$. Using $\varphi(gh) = \varphi(g)\varphi(h)$, we obtain

$$\sum_\eta \eta(gh) A_\eta = \sum_B \left(\sum_{A_\chi A_\psi = B} \chi(g)\psi(h) \right) B.$$

We'll write this entrywise, for $\varphi(gh) = \varphi(g)\varphi(h)$, and using $\eta(gh) = \eta(g)\eta(h)$: this is an equality of coefficients

$$\sum_\eta c_{ij}^\eta \eta(g)\eta(h) = \sum_{\chi,\psi} b_{ij}^{\chi,\psi} \chi(g)\psi(h).$$

The maps $(g,h) \mapsto \eta(g)\eta(h)$ and $(g,h) \mapsto \chi(g)\psi(h)$ are characters of $D \times D$. By linear independence of characters (Dedekind's theorem), the coefficients on both sides of the equality must be equal, i.e., $b_{ij}^{\chi,\psi} = \delta_{\chi,\psi} c_{ij}^\chi$. This proves that the A_χ are orthogonal idempotents.

Finally, we have $1 = \varphi(e) = \sum \chi(e) A_\chi = \sum A_\chi$. Together with the previous paragraph, this proves the claim. Indeed, for $v \in \operatorname{im}(A_\chi)$, we have $v = A_\chi w$, so $\varphi(g)v = \sum_\psi \psi(g) A_\psi A_\chi w = \chi(g) v$; therefore $\operatorname{im}(A_\chi) \subseteq V_\chi$. On the other hand, $V = \bigoplus \operatorname{im}(A_\chi)$, so we must have $\operatorname{im}(A_\chi) = V_\chi$. ■

Dual to characters, we have *one-parameter subgroups*—these play a crucial role in GIT.

Definition 2.9 For an algebraic group G, a **one-parameter subgroup (1-psg)** is a homomorphism $\lambda : \mathbb{G}_m \to G$. Write $\mathrm{Y}(G)$ for the group $\mathrm{Hom}_{\text{alg. gp.}}(\mathbb{G}_m, G)$, with pointwise multiplication.

Note that $\mathrm{Y}(G)$ is *not* necessarily commutative. However, there is always a pairing

$$\mathrm{X}(G) \times \mathrm{Y}(G) \to \mathbb{Z}$$

given by $(\chi, \lambda) \mapsto \chi \circ \lambda \in \mathrm{Hom}_{\text{alg. gp.}}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$.

Exercise 2.10 When $G = T$ is a torus, show that this is a perfect pairing, i.e., it identifies $\mathrm{Y}(T)$ with $\mathrm{X}(T)^\vee = \mathrm{Hom}_\mathbb{Z}(\mathrm{X}(T), \mathbb{Z})$. Is this true more generally when $G = D$ is diagonalizable? *(Answer: Yes, in the sense $\mathrm{Y}(D) = \mathrm{X}(D)^\vee$; note that any 1-psg must have image in the connected component D°, so there is no torsion in $\mathrm{Y}(D)$.)*

3 Reductive groups

The groups with well-behaved invariant theory, for which GIT works best, are the *reductive groups*. Over an algebraically closed field, they're (essentially) classified by Cartan-Killing.

Here I just give definitions and examples, without proof.

3.1 Jordan decomposition

For a linear algebraic group G, an element $x \in G$ is **semisimple** if there is a faithful representation $\rho : G \to GL_n$ such that $\rho(x)$ is diagonal. An element x is **unipotent** if there is a ρ such that $\rho(x) \in \mathbb{U}_n$ is strictly upper-triangular.

Proposition 3.1 *For any $x \in G$, there are unique elements $x_s, x_u \in G$ such that $x = x_s x_u = x_u x_s$, with x_s semisimple and x_u unipotent.*

Moreover, any homomorphism $\varphi : G \to H$ preserves semisimple and unipotent parts.

Reference: [H, §15.3].

3.2 Unipotent and solvable groups

Definition 3.2 A LAG G is **unipotent** if all elements $x \in G$ are unipotent.

The *commutator subgroup* $(G,G) \subseteq G$ is the group generated by all elements $ghg^{-1}h^{-1}$, for $g,h \in G$. It is a closed subgroup [H, ??].

Definition 3.3 The group G is **solvable** if the series

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots ,$$

with $G_i = (G_{i-1}, G_{i-1})$, terminates in the trivial group $\{e\}$.

Example 3.4 The group $\mathbb{U}_n \subset GL_n$ is unipotent, essentially by definition. It is also solvable; the filtration has the first i superdiagonals equal to 0. E.g.,

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \supseteq \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \supseteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 3.5 The group $\mathbb{B}_n \subset GL_n$ is solvable, since $(\mathbb{B}_n, \mathbb{B}_n) = \mathbb{U}_n$.

Clearly any subgroup of a solvable group is solvable, and similarly for unipotent groups.

Also, every unipotent group is solvable. This follows from the fact that $\mathbb{U}_n$ is, together with:

Proposition 3.6 ("Lie-Kolchin") *If G is unipotent, then for every representation $\rho : G \to GL(V)$, there is a basis of V such that $\rho(G) \subseteq \mathbb{U}_n$.*

If G is solvable, then for every representation $\rho : G \to GL(V)$, there is a basis of V such that $\rho(G) \subseteq \mathbb{B}_n$.

3.3 Borel subgroups

As usual, G is a LAG.

Definition 3.7 A **Borel subgroup** of G is a maximal connected closed solvable subgroup.

For example, $\mathbb{B}_n$ is a Borel subgroup in GL_n.

Theorem 3.8 *All Borel subgroups of G are conjugate in G.*

Reference: [H, §21.3]. The proof uses: (1) $G/B \cong \{\text{Borel subgroups}\}$ is a projective variety, and (2) the *Borel fixed point theorem*, which says that when a solvable group acts on a projective variety, there is always a fixed point.

Corollary 3.9 *All maximal tori in G are conjugate.*

3.4 Semisimple and reductive groups

Assume G is a nontrivial connected LAG.

Definition 3.10 The **radical** of G is the maximal connected *normal* solvable subgroup, $R(G)$. The **unipotent radical** is the maximal connected normal unipotent subgroup, $R_u(G)$.

(These are unique, see [H, §19.5].)

Example 3.11 We have $R(GL_n) = \{\text{scalar matrices}\} \cong k^*$. (For maximality, consider the sequence $1 \to k^* \to GL_n \to PGL_n \to 1$, noting that PGL_n is simple.)

Example 3.12 For

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} \subseteq GL_4,$$

we have

$$R_u(P) = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Definition 3.13 The group G is **semisimple** if $R(G) = \{e\}$. It is **reductive** if $R_u(G) = \{e\}$.

Example 3.14 The groups SL_n, PGL_n, and $SL_n \times SL_m$ are semisimple. The groups GL_n and T are reductive.

Remark 3.15 (1) Semisimple implies reductive, since $R_u \subseteq R$.

(2) If G is semisimple, then its center $Z(G)$ is finite. (Otherwise the connected component $Z(G)^\circ$ would be a nontrivial solvable group.)

(3) If G is reductive, then $Z(G)^\circ = R(G)$ is a torus, and (G, G) is semisimple.

(4) For any (connected) G, the quotient $G/R(G)$ is semisimple, and $G/R_u(G)$ is reductive.

Example 3.16 GL_n is reductive, and $(GL_n, GL_n) = SL_n$ and $GL_n/k^* = PGL_n$ are (semi)simple.

3.5 Classification

I won't be able to describe the Cartan-Killing classification here, but its existence is worth mentioning. Up to finite quotient, semisimple groups are products of *simple* groups, and these are classified (over an algebraically closed field). There are four infinite families—represented by SL_n, SO_{2n+1}, Sp_{2n}, and SO_{2n}—and five exceptional types—G_2, F_4, E_6, E_7, E_8.

4 Actions on varieties: some examples

Here are a couple (counter)examples of algebraic group actions on varieties, worth keeping in mind.

Example 4.1 If a torus T acts on a nonsingular projective variety with isolated fixed points, then at least one fixed point has a T-invariant open affine neighborhood, in fact isomorphic to affine space (Bialynicki-Birula).

On the other hand, if the variety X is singular, this may fail. For example, consider $T = k^*$ acting on a nodal rational curve; there is only one T-fixed point, and any T-invariant neighborhood must be the whole (projective) curve. (Incidentally, this implies that the nodal curve cannot be embedded equivariantly in any smooth projective T-variety.)

Example 4.2 If a unipotent group acts on a nonsingular projective variety with isolated fixed points, there may still not be invariant affine open neighborhoods. For example, consider $\mathbb{U}_2 \subset GL_2$ acting on $\mathbb{P}^1 = \mathbb{P}(k^2)$ by the standard action. The point $[1, 0]$ is fixed, but the only invariant neighborhood is all of $\mathbb{P}^1$.

References

[B] A. Borel, *Linear Algebraic Groups*, Springer, 1991.

[H] J. Humphreys, *Linear Algebraic Groups*, Springer, 1981.

[S] T. A. Springer, *Linear Algebraic Groups*, second edition, Birkhäuser, 1998.

Orbit lemma: If a linear algebraic group G acts on a quasi-projective variety X, then (i) every orbit is open in its closure, (i) orbits of minimal dimension are closed and, in particular, (iii) closed orbits exist. Borel Fixed Point Theorem: Any action of a connected solvable group G on a projective variety X has a xed point. Theorem: Any two Borel subgroups of a linear algebraic group are conjugate. Proposition 13: Borel subgroups are parabolic.Â Theorem: Let G be a connected linear algebraic group, B âŠ† G a Borel subgroup and T âŠ† G a maximal torus. Then G is covered by all Borel subgroups, and Gs is covered by all maximal tori: G = gâˆˆG gBgâˆ'1 and Gs = gâˆˆG gT gâˆ'1. Proposition 16: Let G be a connected linear algebraic group and T âŠ† G a torus. Then CG(T) is connected. Linear algebraic groups are matrix groups dened by polynomials; a typi-cal example is the group SLn of matrices of determinant one. The theory of algebraic groups was inspired by the earlier theory of Lie groups, and the classication of algebraic groups and the deeper understanding of their struc-ture was one of the important achievements of last century, mainly led by A. Borel, C. Chevalley and J. Tits.Â I have chosen the classication of reductive linear algebraic groups over algebraically closed elds as the ultimate goal in this course. Of course, there is much to do beyond this â€”in some sense, the interesting things only start happening when we leave the world of algebraically closed eldsâ€" but already reaching this point is quite challenging. A homomorphism of algebraic groups is a group homomorphism that is also a map of varieties. In schemey language, another way to say this is that the functor hG : Schemes â†' Sets factors through Groups. Denition 1.2 A linear algebraic group is an ane variety that is an algebraic group. Example 1.3 The multiplicative group Gm = kâˆ— = Spec k[x, xâˆ'1] is an algebraic group. 1. The coordinate ring k[G] of a linear algebraic group is a (commutative) Hopf algebra: it comes with maps. Î´ : k[G] â†' k[G] âŠ— k[G] = k[G Ã— G] c : k[G] â†' k[G](antipode), Ç« : k[G] â†' k(counit), (comultiplication) This is short term course material on Linear Algebra. This helps is quick review of the concepts. This material helps both students and communication/signal processing engineers.Â By using the Gauss-Lobatto integration formula, these 3 singular integral equations are converted to a system of linear algebraic equations, which are solved numerically. Read more. Book.