MORAVA K-THEORIES OF CLASSIFYING SPACES OF
COMPACT LIE GROUPS

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Abstract. This is the abstract for the talk I will give in the 54th Topology
Symposium in August 6, 2007 at the University of Aizu. I will talk about the
computational aspect of cohomology and generalized cohomology theories of
classifying spaces of connected compact Lie groups.

1. Introduction

Let \( p \) be a prime number. Let \( G \) be a connected compact Lie group and let us
denote by \( BG \) its classifying space. We say \( G \) has \( p \)-torsion if and only if \( H_*(G; \mathbb{Z}) \)
has \( p \)-torsion. In the case \( G \) has no \( p \)-torsion, the mod \( p \) cohomology of \( BG \) is
well-known. However, in the case \( G \) has \( p \)-torsion, the computation of the mod \( p \)
cohomology is not an easy task. I refer the reader for the book of Mimura and Toda
[9] for the detailed account on the cohomology of classifying spaces of compact Lie
groups.

I will give the current state of computation of the mod \( p \) cohomology theory,
Brown-Peterson cohomology and Morava \( K \)-theories of classifying spaces of some
connected compact Lie groups. The coefficient ring of Brown-Peterson cohomology
is

\[
BP^* = \mathbb{Z}[[v_1, v_2, \ldots, v_n, \ldots]],
\]

where \( \deg v_n = -2(p^n - 1) \) and the coefficient ring of the Morava \( K \)-theory \( K(n) \)
is

\[
K(n)^* = \mathbb{Z}/p[v_n, v_n^{-1}].
\]

I will describe the results of the joint work with Yagita [2] and some other results
obtained after writing of [2]. One of the explicit computational results is as follows:

Theorem 1.1. For \( p = 2 \), \( G = G_2 \), the Morava \( K \)-theory \( K(n)^*\text{`(BG_2)`} \) of the
classifying space of the exceptional Lie group \( G_2 \) is given by

\[
K(n)^* \otimes_{BP^*}(BG_2).
\]

As an \( grBP^* \)-module, we have that

\[
grBP^*(BG_2)
\]

is isomorphic to

\[
grBP^*[[y_8, y_{12}, y_{14}]]/(y_{14}p_0, y_{14}p-2, y_{14}p-6) \bigoplus grBP^*[[y_8, y_{12}, y_{14}]]\{w_4\},
\]

where the index indicates the degree.

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The computation of the Brown-Peterson cohomology above is done by Kono and Yagita in [6] and the result above is conjectured in [6].

**Theorem 1.2.** For \((G, p) = (G_2, 2), (E_6, 2), (F_4, 3), (E_7, 3), (E_8, 5)\) and \((PU(pm), p)\) where \(p\) fro, the \(E_2\)-term of the Adams spectral sequence for \(P(n)^*(BG)\) has no odd degree elements and it collapses at the \(E_2\)-level. In particular, we have

\[ K(n)^*(BG) = K(n)^* \otimes_{BP^*} BP^*(BG) \]

and the Morava \(K\)-theory \(K(n)^*(BG)\) has no odd degree elements.

Before we begin to deal with the computation, we would like to mention the interpretation of \(K\)-theory in terms of representation theory. Recall the following theorem.

**Theorem 1.3** (Atiyah-Segal). There is an isomorphism

\[ R(G)^\wedge \rightarrow K(BG), \]

where \(R(G)^\wedge\) is the completion of the complex representation ring of \(G\) with respect to the argumentation ideal.

Also, Morava \(K(0)\)-theory is the ordinary cohomology with coefficient in the rational numbers \(\mathbb{Q}\) or its \(p\)-completion \(\mathbb{Q}_p\). Thus, through deRham theory, it is related to differential forms. Morava \(K(1)\)-theory is related to the \(p\)-localization of the complex \(K\)-theory and to vector bundles. Morava \(K(2)\)-theory is related to the elliptic cohomology. Many mathematicians dream of geometric and/or representation theoretical interpretation of Morava \(K\)-theories and related cohomology theories, e.g. elliptic cohomology, complex cobordism theory. We hope the computation of Morava \(K\)-theories of classifying spaces might shed some light on such geometric and/or representation theoretical interpretation.

2. Ordinary cohomology theory

As we already mentioned, when \(G\) has no \(p\)-torsion, we have a satisfactory result:

**Theorem 2.1.** If \(G\) has no \(p\)-torsion, then the induced homomorphism

\[ H^*(BG; \mathbb{Z}/p) \rightarrow H^*(BT; \mathbb{Z}/p) \]

is a monomorphism. If \(p\) is an odd prime, then the image of the above homomorphism is the ring of invariants of the Weyl group \(W\),

\[ H^*(BG; \mathbb{Z}/p) = H^*(BT; \mathbb{Z}/p)^W = \mathbb{Z}/p[y_1, \ldots, y_n] \]

where \(\deg y_1 \cdots \deg y_n = 2^n |W|\) and \(\deg y_1, \ldots, \deg y_n\) are even. In particular, the mod \(p\) cohomology of \(BG\) is a polynomial algebra over \(\mathbb{Z}/p\).

There are compact Lie groups \(G\) with \(p\)-torsion. The following is the list of \(p\)-torsions of simply connected, simple compact Lie groups.
Most of the mod $p$ cohomology theories of classifying spaces of the above Lie groups as graded $\mathbb{Z}/p$-modules are computed. Only the cases $p = 2$, $G = E_8$ and $p = 3$, $G = E_8$ remain unsolved, although some of details are not yet in the literature.

There are important examples of connected compact Lie groups with $p$-torsion. Among those are the projective classical groups, such as the projective unitary group $PU(n)$ which is the quotient of the unitary group $U(n)$ by its center $S^1$. The ordinary cohomology theories of projective classical groups seem to be difficult to compute. Among the nontrivial cases, only the cases $p = 2$, $G = PO(4n + 2)$, $PU(4n + 2)$, $Sp(4n + 2)$ and $p = 3$, $G = PU(3)$ were computed in the 20th century (See [3], [4], [5]). The case $p > 3$, $G = PU(p)$ are computed in [2], [11], recently. The computation of $H^*(PU(p^n); \mathbb{Z}/p)$ ($n \geq 2$) seems to be difficult.

When $G$ has $p$-torsion, there are odd degree elements in the mod $p$ cohomology of $BG$, so that the induced homomorphism

$$H^*(BG; \mathbb{Z}/p) \to H^*(BT; \mathbb{Z}/p)$$

is no longer a monomorphism. Replacing the maximal torus by elementary abelian $p$-subgroups $A$’s, Quillen proved the following theorem.

**Theorem 2.2.** There is an $F$-isomorphism

$$H^*(BG; \mathbb{Z}/p) \to \lim_\leftarrow H^*(BA; \mathbb{Z}/p).$$

$F$-isomorphism implies that a power of $x \in \lim_\leftarrow H^*(BA; \mathbb{Z}/p)$ is in the image of this homomorphism and each element in the kernel of this homomorphism is nilpotent.

The cohomology of elementary abelian $p$-subgroups is not only useful in the computation of the mod $p$ cohomology of classifying spaces but also important to our understanding it. So, we hope the following conjecture to be true.

**Conjecture 2.3.** For $p > 2$, the induced homomorphism

$$H^*(BG; \mathbb{Z}/p) \to \prod_A H^*(BA; \mathbb{Z}/p)$$

is a monomorphism where $A$ ranges over the conjugacy classes of elementary abelian $p$-subgroups of $G$.

In the case $p = 2$, the above conjecture does not hold. See Kono-Yagita [6].

3. *MORAVA K-THEORY AND BROWN-PETerson COHOMOLOGY THEORY*

The computation of generalized cohomology theories is easy when $G$ has no $p$-torsion and the coefficient ring of the generalized cohomology theory has no odd

<table>
<thead>
<tr>
<th>Lie group</th>
<th>2-torsion</th>
<th>3-torsion</th>
<th>5-torsion</th>
<th>$p$-torsion ($p &gt; 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(n)$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$Sp(n)$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$Spin(n)$</td>
<td>$\circ$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\circ$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\circ$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>
degree elements. Since the ordinary cohomology of $BG$ has no odd degree elements, the $E_2$-term
$$E_2^{p,q} = H^p(BG; E^q)$$
of the Atiyah-Hirzeburch spectral sequence converging to the generalized cohomology theory $E^* (BG)$ collapses at the $E_2$-level and we have the following proposition.

**Proposition 3.1.** The Brown-Peterson cohomology $BP^*(BG)$ is isomorphic to
$$BP^* \otimes_{\mathbb{Z}/p} H^*(BG; \mathbb{Z}/(p)).$$
The Morava K-theory $K(n)^*(BG)$ is isomorphic to
$$K(n)^* \otimes_{\mathbb{Z}/p} H^*(BG; \mathbb{Z}/p).$$

If $G$ has $p$-torsion, then the mod $p$ cohomology has an odd degree nonzero element. Our results seem to support the following conjecture.

**Conjecture 3.2.** The Brown-Peterson cohomology of the classifying space of a connected compact Lie group has no odd degree elements.

Many mathematicians believed that this conjecture should be true for all compact Lie groups including all finite groups. But a counterexample was constructed by Kriz in [7], [8]. We do not know any counterexample in the case of connected compact Lie groups.

There is a generalized cohomology theory $P(n)$ for $n \geq 0$. $P(0)$ is $BP$ or its $p$-completion. We compute the $P(n)$-cohomology in order to compute the Morava K-theories.

**Theorem 3.3.** If the induced homomorphism
$$p : BP^*(X) \to P(n)^*(X)$$
is an epimorphism for all $n \geq 0$, then the following hold: for all $n \geq 0$,
$$P(n)^*(BG) \cong P(n)^* \otimes_{BP^*} BP^*(BG)$$
and
$$K(n)^*(BG) \cong K(n)^* \otimes_{BP^*} BP^*(BG).$$
So, if we would like to show that
$$K(n)^*(X) \cong K(n) \otimes_{BP^*} BP^*(X)$$
for all $n$, it suffices to show that
$$BP^*(X) \to P(n)^*(X)$$
is an epimorphism for all $n$.

In the case $p = 2$, we have the following results for simply connected simple Lie groups.

<table>
<thead>
<tr>
<th>$(p = 2)$</th>
<th>$G$</th>
<th>BP</th>
<th>Morava K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_2$</td>
<td>∅</td>
<td>∅</td>
<td>??</td>
</tr>
<tr>
<td>$F_4$</td>
<td>∅</td>
<td>??</td>
<td>??</td>
</tr>
<tr>
<td>$E_6$</td>
<td>∅</td>
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<td>??</td>
</tr>
<tr>
<td>$E_7$</td>
<td>??</td>
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<td>??</td>
</tr>
<tr>
<td>$E_8$</td>
<td>??</td>
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<td>??</td>
</tr>
</tbody>
</table>
The computation of the Brown-Peterson cohomology above is due to Kono and Yagita in [6]. The computation of Morava K-theories is a new result. Kono and Yagita compute the Brown-Peterson cohomology of $B\text{Spin}(n)$ for $n \leq 10$ and $B\text{PU}(4n+2)$, $B\text{Sp}(4n+2)$, too. Wilson compute the Brown-Peterson cohomology of $BO(n)$ in [12] and Kono and Yagita compute their Morava K-theories in [6]. Recently, Inoue and Yagita compute the Brown-Peterson cohomology and Morava K-theories of $BSO(n)$ for $n \geq 2$ in [1]. In the case $p = 3$, we have the following results for simply connected simple Lie groups.

\[
\begin{array}{c|c|c|c}
(p = 3) & G & BP & \text{Morava } K \\
\hline
F_4 & \circ & \circ \\
E_6 & \circ & ?? \\
E_7 & \circ & \circ \\
E_8 & ?? & ?? \\
\end{array}
\]

The computation of $p = 3$, $G = F_4$, $PU(3)$ is done in [6]. In the case $p = 3$, $G = E_6$, in [2], we computed the Brown-Peterson cohomology of $BE_6$, however, we could not conclude

\[
K(n)^*(BE_6) = K(n)^* \otimes_{BP} BP^*(BE_6).
\]

It remains to be an open problem. In the case $p = 5$, we have the following result for simply connected simple Lie groups.

\[
\begin{array}{c|c|c|c}
(p = 5) & G & BP & \text{Morava } K \\
\hline
E_8 & \circ & \circ \\
\end{array}
\]

4. Adams spectral sequence

In [6], they use the Atiyah-Hirzebruch spectral sequence in order to compute the Brown-Peterson cohomology. The Atiyah-Hirzebruch spectral sequence in [6] does not collapse at the $E_2$-level. In [2], we use the Adams spectral sequence in order to compute $P(n)$-cohomology and show that the assumption in Theorem 3.3 holds. The Adams spectral sequence is an spectral sequence converging to $P(n)$ cohomology whose $E_2$-term is given by

\[
\text{Ext}_A(H^*(P(n); \mathbb{Z}/p), H^*(BG; \mathbb{Z}/p)) = \text{Ext}_E(Z/p, H^*(BG; \mathbb{Z}/p)),
\]

where $A$ is the mod $p$ Steenrod algebra and $E_n$ is the subalgebra generated by Milnor operations $Q_n, Q_{n+1}, \ldots$. The $E_2$-term could be computed by taking the (co)homology of the (co)chain complex

\[
d_1 : gr P(n)^* \hat{\otimes} H^*(BG; \mathbb{Z}/p) \to gr P(n)^* \hat{\otimes} H^*(BG; \mathbb{Z}/p)
\]

where

\[
gr P(n)^* = \mathbb{Z}/p[v_n, v_{n+1}, \ldots]
\]

and

\[
d_1(v \otimes x) = \sum_{k=n}^{\infty} vv_k \otimes Q_kx.
\]

We compute this (co)chain complex. In this talk, we deal with the case $p = 2$, $G = G_2$, $E_6$ and give an outline of the computation.

First, we deal with the case $p = 2$, $G = G_2$. Let us recall the mod 2 cohomology of the classifying space of the exceptional Lie group $G_2$. There is a non-toral elementary abelian 2-subgroup of rank 3, say $A$, in $G_2$. The induced homomorphism

\[
H^*(BG_2; \mathbb{Z}/2) \to H^*(BA; \mathbb{Z}/2)
\]
Proposition 4.2. Let $\text{generated by}$

Let $\text{consider}$

Let $\text{integral domain.}$ Let $\text{corresponds to}$

the following proposition completes the proof of Theorem 1.1. The element

Since $\text{has no odd degree elements and since}$

Proposition 4.1. There exit $f_1, f_2, f_3 \in grP(n)^* \otimes R$ such that

where $\text{deg } f_1 = 0,$ $\text{deg } f_2 = -2,$ $\text{deg } f_3 = -6.$ Moreover, $f_1, f_2, f_3$ is a regular sequence in $grP(n)^* \otimes R.$

Since $\text{R has no odd degree elements and since}$

the following proposition completes the proof of Theorem 1.1. The element

corresponds to $w_4$ in Theorem 1.1.

Proposition 4.2. Let $R$ be a graded algebra over $\mathbb{Z}/2$ and suppose that $R$ is an integral domain. Let $f_1, \ldots, f_n$ be elements in $R.$ Let $z_0$ be an element in $R$ and consider $R$-submodules $C_k$ of

generated by

where $\Delta(\beta_1, \ldots, \beta_n)$ is a simple system of generators, $1 \leq i_1 < i_2 < \cdots < i_k \leq n,$ $k > 0$ and $C_0 = R.$ Consider the differential $d$ given by $d(\beta_k) = z_0 f_k$ for $k = 1, \ldots, n$ and by $d(xy) = d(x)y + xd(y).$ If $f_1, \ldots, f_n$ is a regular sequence, then the homology of $(C, d)$ is given by

and

for $k > 0.$

Next, we deal with the case $p = 2,$ $G = E_6.$ The mod 2 cohomology $H^*(BE_6; \mathbb{Z}/2)$ is generated by $y_4, y_6, y_7, y_{10}, y_{18}, y_{32}, y_{48}$ and $y_{34}$ with the relations

$y_7y_{10} = 0, \quad y_7y_{18} = 0, \quad y_7y_{34} = 0, \quad y_{34}^2 = y_{10}^2y_{48} + y_{18}^2y_{32} + \text{lower terms},$
where the index indicates the degree. There is an elementary abelian 2-subgroup $A$ of rank 5 in $F_4 \subset E_6$. The induced homomorphism 

$$H^*(BE_4; \mathbb{Z}/2) \to H^*(BA; \mathbb{Z}/2)$$

is not a monomorphism. The image of this induced homomorphism is isomorphic to 

\[ H^*(BG_2; \mathbb{Z}/2) \otimes \mathbb{Z}/2[y_{16}^2, y_{24}^2] \]

and the action of Milnor operations on $y_{16}^2, y_{24}^2$ is trivial. The kernel of this induced homomorphism is the ideal generated by $y_{10}, y_{18}, y_{34}$ and it has no odd degree elements. So, by investigating the long exact sequence of Ext groups induced by the short exact sequence 

\[ 0 \to (y_{10}, y_{18}, y_{34}) \to H^*(BE_6; \mathbb{Z}/2) \to H^*(BG_2; \mathbb{Z}/2) \otimes \mathbb{Z}/2[y_{16}^2, y_{24}^2] \to 0 \]

of $E_\infty$-modules, we have the collapsing of the Adams spectral sequence.

5. Beyond connected compact Lie groups

There seem to be several directions to extend the homotopy theory of classifying spaces of connected compact Lie groups. One of them is the study of the cohomology of finite Chevalley groups and free loop spaces of classifying spaces. I would like to end this talk with this.

For a connected compact Lie group $G$, there is a complexification $G(\mathbb{C})$. $G(\mathbb{C})$ is a connected reductive complex algebraic group and there is a reductive integral group scheme $G_{\mathbb{Z}}$ such that $G_{\mathbb{Z}}(\mathbb{C}) = G(\mathbb{C})$ as an algebraic group. Replacing $\mathbb{C}$ by $\mathbb{F}_q$, we have finite Chevalley group $G(\mathbb{F}_q)$. By Friedlander-Quillen theory, if $q$ is a power of $p$ and if $\ell$ is a prime number not equal to $p$, then the mod $\ell$ cohomology of the finite Chevalley group is isomorphic to the mod $\ell$ cohomology of the following pull-back $F_{\phi^q}$:

\[
\begin{array}{ccc}
F_{\phi^q} & \to & BG^\wedge \\
\downarrow & & \downarrow 1 \times 1 \\
BG^\wedge & \xrightarrow{1 \times \phi^q} & BG^\wedge \times BG^\wedge,
\end{array}
\]

where $\phi^q$ is the Frobenius map and $BG^\wedge$ is the Bousfield-Kan $\mathbb{Z}/\ell$-completion. If $q - 1 \equiv 0 \mod \ell$, the induced homomorphism

\[ 1 - \phi^{q*} : H^*(BG^\wedge; \mathbb{Z}/\ell) \to H^*(BG^\wedge; \mathbb{Z}/\ell) \]

is zero and the Eilenberg-Moore spectral sequence for $H^*(F_{\phi^q}; \mathbb{Z}/\ell)$ has the same $E_2$-term with the Eilenberg-Moore spectral sequence for $H^*(LBG; \mathbb{Z}/\ell)$ where $LBG$ is the free loop space of $BG$ and it is the pull-back of the following diagram.

\[
\begin{array}{ccc}
\mathcal{L}BG & \to & BG \\
\downarrow & & \downarrow 1 \times 1 \\
BG & \xrightarrow{1 \times 1} & BG \times BG.
\end{array}
\]

If $G$ has no $\ell$-torsion, both Eilenberg-Moore spectral sequences collapse at the $E_2$-level and we have the same mod $\ell$ cohomology for finite Chevalley groups $BG(\mathbb{F}_q)$ and free loop space $LBG$. Thus, we have the same mod $\ell$ cohomology.
On the other hand, the induced homomorphism
\[ 1 - \phi^*: K(n)^*(BG^\ell) \to K(n)^*(BG) \]
is not zero, where \( K(n)^* = \mathbb{Z}/\ell[v_n, v_n^{-1}] \). So the \( E_2 \)-term of the Eilenberg-Moore spectral sequence for \( K(n)^*(BG(F_q)) \) differs from the one for \( K(n)^*(LBG) \). Again, when \( G \) has no \( \ell \)-torsion, we have a satisfactory answer for Morava \( K \)-theories of finite Chevalley groups.

**Theorem 5.1** (Tanabe [10]). If \( G \) has no \( \ell \)-torsion, then \( K(n)^*(BG(F_q)) \) has no odd degree elements.

When \( G \) has an \( \ell \)-torsion, we know little on the mod \( \ell \) cohomology, Brown-Peterson cohomology and Morava \( K \)-theories of classifying spaces of finite Chevalley groups \( G(F_q) \) and the free loop space \( LBG \).

### References


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Twisted K-theory of compact Lie groups has been studied by physicists (see e.g. [Br04] [MMS01] [GG04]) as well as by mathematicians, starting with Douglas [Do06]. The computation of the twisted K-groups was extended to Lie groups are not necessarily compact simple and simply connected in [GG04][MR17]. The results for twisted K-theory KEpG, hq, for arbitrary choices of the twist h are already rather complicated and hard to understand. The twisted Morava K-homology of all groups in the Whitehead tower of the orthogonal and unitary groups, and their classifying spaces, with the canonical twist, is isomorphic to the underlying untwisted Morava K-homology. This seems to be an instance of a more general phenomenon, and it does occur even in the case of K-homology. In particular, we discuss the classifying spaces BG that are p-complete for all primes when the groups are certain subgroups of simple Lie groups. A survey of the p-complete groups is included. 55R35; 55P15, 55P60. A p-complete group (see Dwyer-Wilkerson [8]) is a loop space X such that X is Fp-anisiot and that its classifying space BX is Fp-complete (see Andersen-Grodal-Weier sebuahh Møller-Viruel [2] and Dwyer-Wilkerson [11]). We recall that the p-completion of a compact Lie group G is a p-complete group if Π0(G) is a p-complete group. Next, if C(I) denotes the centralizer of a group homomorphism MORAVA K-THEORY OF EILENBERG-MAC LANE SPACES ERIC PETERSON This talk is about a 1980s computation by Ravenel and Wilson of the Morava K-theories of certain Eilenberg-Mac Lane spaces. The most basic situation is q = 1, so that we're studying the classifying space HZ/pj 1 = BZ/pj. Keep in mind that really the only things we know about Morava K-theory are: (1) Its coefficient ring is K = F p n[v± 1], which happens to be a graded field. (2) We also understand its p-series, which is the action of the map CP.sz CP.sz on cohomology. Facts about these objects are best-organized by a tool called Dieudonné theory, which is an analogue of a theory of Lie algebras for p-divisible groups, but I cannot go into that now.